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Cut-off QFT and black hole entropy

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ABSTRACT: We clarify the relationship between black hole entropy and the number of degrees of freedom in the dual QFT with a cut-off. We show that simple gravity arguments predict the correct cut-off procedure.

KEYWORDS: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

As is well known the entropy of a black hole is [1]

$$S_{BH} = \frac{A}{4G}\,, \tag{1.1}$$

where A is the area of the horizon and G is Newton constant. This has led to the idea of holography [2], which claims that the number of degrees of freedom in a gravitational system is proportional to its area and not its volume, and that black hole configurations maximize the entropy for a given energy and charge.

In the context of AdS/CFT [3] this takes a more precise form. There is a connection between a CFT with a cut-off (to be defined later) and portions of AdS [4] . The full CFT lives at the boundary of AdS and is dual to string theory in the whole bulk of AdS. If one considers a CFT with some cut-off then one can think of the cut-off CFT as 'living" on some time-like hyper-surfaces in AdS, and this cut-off CFT is dual to string theory in the inner bulk. Holography then implies that the number of degrees of freedom of the cut-off CFT (to be defined), related to some constant radial hyper-surface, is exactly the area of the hyper-surface in the AdS space divided by $4G$. In [4] some definitions of the number of degrees of freedom of a cut-off CFT were used and it was shown that it is proportional to the area in Planck units but they have not specified the regularization scheme that one should use.

Three quantities; The black hole entropy, the area in Planck units and the number of degrees of freedom in a cutoff CFT need to match to each other. Given the AdS/CFT one should be able to understand why these quantities are equal. Recently the relationship between black hole entropy and the area in Planck units was explained in this context [5]. In this note we would like to give a better understanding of the equality between the black hole entropy, which is the dual QFT thermal entropy and the number of degrees of freedom of the QFT with some cutoff.

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2. Thermality and cut-off

Let us start with a CFT and label the number of states with energy E by $\Omega(E)$. The total number of states in a local quantum field theory without a cut-off

$$\int_{-\infty}^{\infty} \Omega(E)dE \tag{2.1}$$

is of course infinite. At any given energy one could have defined (ad hoc) the number of degrees of freedom to be

$$\ln \Omega(E)\,. \tag{2.2}$$

If one looks at a quantum field theory with some cut-off scheme then one can define a regulated version of (2.1)

$$N(\Lambda) = \int f(\Lambda, E)\Omega(E)dE\,, \tag{2.3}$$

where $f(\Lambda, E)$ is a regulating function defining the regularization scheme. How should one define then the regulated number of degrees of freedom ? Since $\Omega(E)$ is a rapidly increasing function (at least if the number of degrees of freedom is large) and $f(\Lambda, E)$ a rapidly decreasing function of E, then to compute the number of states with this regulating scheme one has to preform a saddle point approximation to find a saddle point $\bar{E}$, and the regulated number of degrees of freedom can then be defined as

$$\ln \Omega(\bar{E})\,. \tag{2.4}$$

On the other hand the black hole is dual to the CFT at some finite temperature. The partition function of the CFT at finite temperature is

$$Z(\beta) = \int e^{-\beta E}\Omega(E)dE\,. \tag{2.5}$$

We can now view equation (2.5) in another manner. $Z(\beta)$ can be viewed as a regulated number of states with a particular regulator(cut-off). Namely $e^{-\beta E}$ is just a cut-off function of the states of the CFT. Then the regulated number of degrees of freedom $\ln \Omega(\bar{E})$ is of course just the definition of the entropy if one views $Z(\beta)$ as a finite temperature partition function. This suggests that the appropriate definition of the cut-off CFT is with an exponential decreasing regulating function. The question is, is this compatible with other properties of the AdS.

3. Gravity side

In this section we will show that the AdS/CFT in the regime of semiclassical supergravity suggests a regularization scheme one should use, which matches the one from the previous section.

3.1 Cut-off and lightcone

The relation between the energy cut-off and the radial coordinate of the hyper-surface on which the cut-off theory is defined, is encoded in the null geodesic equation. A localized object in the bulk is represented by a blob on the boundary. let us denote the size of the blob as L. The lorentzian nature of the boundary theory implies (where t is the boundary time)

$$\frac{dL}{dt} \leq 1\,. \tag{3.1}$$

Let us now imagine the object moving radially on some trajectory where the radial position is changing (and hence the cut-off Λ, where Λ^{-1} is the energy cut-off)), then

$$\frac{dL}{dt} = \frac{dL}{d\Lambda}\frac{d\Lambda}{dt}\,. \tag{3.2}$$

Note that in general $L \geq \Lambda$, although for pointlike objects in the bulk $L = \Lambda$. Now since the blob size changes due to the changing cut-off and in order not to violate equation (3.1) one has

$$\frac{dL}{d\Lambda} \leq 1 \qquad \frac{d\Lambda}{dt} \leq 1\,. \tag{3.3}$$

Since equation (3.1) can be saturated this implies that the lightcone condition on the boundary is related to the lightcone condition in the bulk [6], as Λ is a function of the radial direction (labeled by r). To get equality in (3.1) one needs both equalities in (3.3), in particular this gives

$$\left|\frac{d\Lambda(r)}{dt}\right| = 1\,, \tag{3.4}$$

as a condition of identifying the r dependence of Λ, as a lightcone condition in the bulk.

Let us now look at some examples.

We start with the metric

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dx_i^2 + dz^2)\,. \tag{3.5}$$

The lightcone condition is

$$\frac{dz}{dt} = 1 \tag{3.6}$$

which then gives

$$\Lambda = z\,. \tag{3.7}$$

In another form we can take

$$ds^2 = \frac{U^2}{R^2}(-dt^2 + dx_i^2) + \frac{R^2}{U^2}dU^2\,. \tag{3.8}$$

The lightcone condition is

$$\frac{dU}{dt} = \frac{U^2}{R^2}\,. \tag{3.9}$$

To turn this equation into the form of (3.4) one gets

$$\Lambda = \left|\int R^2\frac{dU}{U^2}\right| = \frac{R^2}{U}\,, \tag{3.10}$$

which agrees with [7]. One can also take the metric for global AdS, of the form

$$ds^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \left(1+\frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \,. \tag{3.11}$$

Here we find

$$\Lambda = l\int \frac{dr}{1+r^2/l^2} = l\left(\arctan\left(\frac{r}{l}\right)+c\right). \tag{3.12}$$

We need to obey the boundary condition $\Lambda(r=\infty)=0$, so we find

$$r = l\tan\left(-\frac{\Lambda}{l}+\frac{\pi}{2}\right). \tag{3.13}$$

Notice that this implies that at $r=0$ the energy cut-off is $2/(l\pi)$ which is appropriate for a theory on a sphere.

3.2 Regulating function

For simplicity we work with the Euclidean metric

$$ds^2 = \frac{l^2}{z^2}(dz^2 + dx_i^2)\,. \tag{3.14}$$

In the semiclassical gravity regime a scalar field in the bulk is related to a scalar field at the boundary as

$$\phi(x,z) = \int dx' G(z,x,x')\phi(x')\,, \tag{3.15}$$

where the kernel depends on the mass of the scalar (or on the conformal dimension of the corresponding CFT operator Δ), and is given by [8]

$$G(z,x,x') \sim \left(\frac{z}{z^2-|x-x'|^2}\right)^{\Delta}\,. \tag{3.16}$$

Since the supergravity fields are dual to sources that couple to operators in the CFT, We can think of equation (3.15) as defining a smeared source and thus a regularization scheme.[1] To see more clearly the content of this scheme we Fourier transform equation (3.15)

$$\phi(\vec{k}) \sim K_{\Delta-d/2}(kz)\phi'(\vec{k})\,, \tag{3.17}$$

where $K_\nu(x)$ is a Bessel function and $k=|\vec{k}|$. The asymptotic behavior of the Bessel function is (up to powers of kz which are not relevant for the saddle point)

$$\sim e^{-kz} \tag{3.18}$$

for all ν. Similar regularization scheme can be found for vector and higher spin fields. The important thing is that they all have the same exponential asymptotic form at large kz. So we see that indeed the AdS/CFT relationship predicts a regulating function of the form

[1] In general bulk operators are not simply related to boundary operators [9] other than in the semiclassical regime.

that is compatible with the relationship to the thermal partition function of the CFT and thus to the black hole entropy[2], with a cutoff

$$z \sim \frac{1}{T}\,. \tag{3.19}$$

To see if this cut-off relationship is correct, all we have to do is to check if the radius size of a black hole at temperature T obeys equation (3.19).

Indeed for AdS_3 we have

$$z_h = \frac{1}{2\pi T} \tag{3.20}$$

and for AdS_5 one gets

$$z_h = \frac{1}{\pi T}\,. \tag{3.21}$$

Note that this does not follow from dimensional analysis since any function of the dimensionless coupling constant could have been involved. Now we see that approximating the regularization function by its asymptotic behavior is valid since in the semiclassical regime $\bar{E}/T \sim S_{bh} \gg 1$.

3.3 Non conformal case

For non conformal theories which are dual to the near horizon geometry of p-branes, one can do a similar analysis. The geometry is given by [10]

$$ds^2 = \frac{U^{(7-p)/2}}{\sqrt{d_p e^2}}(-dt^2 + dx_i^2) + \frac{\sqrt{d_p e^2}}{U^{(7-p)/2}} dU^2 + \sqrt{d_p e^2}\, U^{(p-3)/2} d\Omega_{8-p}^2\,, \tag{3.22}$$

where $e^2 = g_{ym}^2 N$ and $d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma(\frac{7-p}{2})$. The wave equation for a massless scalar in ten dimension is given by (see for example [11])

$$\left[\partial_U(U^{8-p}\partial_U) - \sqrt{d_p e^2}\, k^2 U + l(l+7-p)U^{6-p}\right]\phi = 0\,, \tag{3.23}$$

where $l(l+7-p)$ is some eigenvalue of the laplacian on the sphere S^{8-p}. If we define

$$z = \frac{\sqrt{d_p e^2}}{\frac{5-p}{2} U^{(5-p)/2}}\,. \tag{3.24}$$

Then the appropriate solution takes the form

$$(z)^{(p-7)/(p-5)} K_\nu(zk)\,. \tag{3.25}$$

Since Solution of equation (3.23) controls the behavior of the bulk to boundary Greens function at large kz, we see that the behavior at large kz is again

$$e^{-kz}\,. \tag{3.26}$$

[2]The saddle point is expected to be at $\bar{\mathbf{P}} = 0$ where $\mathbf{P}$ is the momenta

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For this to match the gauge theory thermal entropy one should then have that a black hole of temperature T have a horizon size

$$T \sim \frac{1}{z_h} \tag{3.27}$$

which is indeed the case since for these black holes

$$T = \frac{7-p}{2\pi(5-p)} \frac{1}{z_h}\,. \tag{3.28}$$

To conclude, we see that the thermal entropy relationship with the number of degrees of freedom of a cut-off gauge theory is based on a certain regulating scheme whose functional form (in the large N large t Hooft coupling region) can be predicted from the semiclassical gravity.

The arguments in this paper suggests that $T = 1/z_h$, but infact one finds that for the black holes $z_h < 1/T$ (i.e. the cut-off procedure suggests that there are more degrees of freedom than the BH entropy). The meaning of this is not clear. It could be that the naive view of how the cut-off procedure is done is not accurate although it does give the correct dependence on the parameters.

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